

ON STRICTLY CYCLIC ALGEBRAS, $\mathcal{P}$ -ALGEBRAS AND REFLEXIVE OPERATORS

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ABSTRACT. An operator algebra $\mathfrak{A} \subset \mathcal{L}(\mathcal{X})$ (the algebra of all operators in a Banach space $\mathcal{X}$ over the complex field $\mathbb{C}$) is called a "strictly cyclic algebra" (s.c.a.) if there exists a vector $x_0 \in \mathcal{X}$ such that $\mathfrak{A}(x_0) = \{Ax_0 : A \in \mathfrak{A}\} = \mathcal{X}$; x_0 is called a "strictly cyclic vector" for $\mathfrak{A}$. If, moreover, x_0 separates elements of $\mathfrak{A}$ (i.e., if $A \in \mathfrak{A}$ and $Ax_0 = 0$, then $A = 0$), then $\mathfrak{A}$ is called a "separated s.c.a."

$\mathfrak{A}$ is a $\mathcal{P}$ -algebra if, given $x_1, \dots, x_n \in \mathcal{X}$, there exists $x_0 \in \mathcal{X}$ such that $\|Ax_j\| \leq \|Ax_0\|$, for all $A \in \mathfrak{A}$ and for $j = 1, \dots, n$. Among other results, it is shown that if the commutant $\mathfrak{A}'$ of the algebra $\mathfrak{A}$ is an s.c.a., then $\mathfrak{A}$ is a $\mathcal{P}$ -algebra and the strong and the uniform operator topology coincide on $\mathfrak{A}$; these results are specialized for the case when $\mathfrak{A}$ and $\mathfrak{A}'$ are separated s.c.a.'s. (Here, and in what follows, *algebra* means *strongly closed subalgebra* on $\mathcal{L}(\mathcal{X})$ containing the identity I on $\mathcal{X}$.)

In the second part of the paper, it is shown that a large class of bilateral weighted shifts (which includes all the invertible ones) in a Hilbert space are reflexive. The result is used to show that "reflexivity" is neither a "restriction property" nor a "quotient property."

Recall that an algebra $\mathfrak{A}$ is called *reflexive* if, whenever $T \in \mathcal{L}(\mathcal{X})$ and the lattice of invariant subspaces of T contains the corresponding lattice of $\mathfrak{A}$, then $T \in \mathfrak{A}$. (?)

1. A strictly cyclic algebra and its commutant. Let $\mathfrak{A}$ be a strictly cyclic algebra on $\mathcal{X}$, with strictly cyclic vector x_0 . We have a commutative diagram:

$$\begin{array}{ccc} \mathfrak{A} & \xrightarrow{S} & \mathfrak{A}' \\ \pi \downarrow & \nearrow \bar{S} & \\ \mathfrak{A}/\ker S & & \end{array}$$

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(2) After this paper was written, the authors received the preprint *Notes on strictly cyclic operator algebras* by Mary R. Embry. In that paper, M.R. Embry found independent proofs of Theorem 1(i) and Theorem 3 below.

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where $S(A) = Ax_0$, $\ker S$ is a closed left ideal in $\mathfrak{A}$, π is the canonical projection onto the quotient space and $\bar{S}$ ($S = \bar{S} \cdot \pi$) is the quotient map. Clearly, S is continuous, linear and onto, π has the same properties and $\bar{S}$ is an isomorphism of Banach spaces. Moreover, π (and therefore S) is an open map.

Theorem 1. *Let $\mathfrak{A}$ be a subalgebra of $\mathcal{L}(\mathcal{X})$ and let $\mathfrak{A}' = \{T \in \mathcal{L}(\mathcal{X}): TA = AT, \text{ for all } A \in \mathfrak{A}\}$ be the commutant of $\mathfrak{A}$. Assume that $\mathfrak{A}'$ has an s.c. vector x_0 ; then*

- (i) $\mathfrak{A}$ has the property $(\mathcal{P})$ (i.e., $\mathfrak{A}$ is a $\mathcal{P}$ -algebra);
- (ii) the strong and the uniform operator topologies coincide on $\mathfrak{A}$.

Proof. (i) By hypothesis, $\mathcal{X} = \mathfrak{A}'x_0$. Thus, given $x_0, \dots, x_n \in \mathcal{X}$, there exist $A_1, \dots, A_n \in \mathfrak{A}'$ such that $x_j = A_j x_0$ for $j = 1, \dots, n$. Moreover, since S is an open map, the A_j 's can be chosen in such a way that $\|A_j\| \leq C\|x_j\|$, where C is a positive constant, independent of j (see [5, Lemma 1]).

Thus, if $B \in \mathfrak{A}$, then

$$\|Bx_j\| = \|BA_j x_0\| = \|A_j Bx_0\| \leq \|A_j\| \cdot \|Bx_0\| \leq \|B(Mx_0)\|,$$

where $M = C \max\{\|x_j\|: j = 1, \dots, n\}$.

Hence, $\mathfrak{A}$ is a $\mathcal{P}$ -algebra.

(ii) Let $\{A_\lambda: \lambda \in \Lambda, \text{ a directed set}\}$ be a net in $\mathfrak{A}$ converging strongly to A ($\in \mathfrak{A}$). Let $x = Bx_0$, $B \in \mathfrak{A}'$ be chosen s.t. $\|x\| \leq 1$ and $\|B\| \leq C$. Then

$$\|(A_\lambda - A)x\| = \|(A_\lambda - A)Bx_0\| = \|B(A_\lambda - A)x_0\| \leq C\|(A_\lambda - A)x_0\|.$$

Therefore

$$\lim_{\lambda \in \Lambda} \|A_\lambda - A\| \leq C \lim_{\lambda \in \Lambda} \|(A_\lambda - A)x_0\| = 0;$$

i.e., strongly convergent nets are actually norm-convergent. It follows immediately that the strong and the uniform topologies coincide on $\mathfrak{A}$.

Now, most of the results of [6, §3] can be "translated" to these algebras (or sums of them). Consider, in particular, the four algebras associated with an operator $T \in \mathcal{L}(\mathcal{X})$ defined in [6]; i.e., the algebras $\mathfrak{A}_T$ and $\mathfrak{A}_T^a$ generated by the polynomials and by the rational functions (with poles outside of the spectrum) in T , resp., and the commutant $\mathfrak{A}'_T$ and the double commutant $\mathfrak{A}''_T$ of T . Then $\mathfrak{A}_T \subset \mathfrak{A}_T^a \subset \mathfrak{A}'_T \subset \mathfrak{A}''_T$, and the corresponding lattices of invariant subspaces satisfy the reverse inclusions. The above mentioned results yield the following

Corollary 1. *If $T \in \mathcal{L}(\mathcal{X})$ and $\mathfrak{A}'_T$ is an s.c.a., then any two of the algebras $\mathfrak{A}_T$, $\mathfrak{A}_T^a$, $\mathfrak{A}''_T$ and $\mathfrak{A}'_T$ are equal if and only if the corresponding lattices of invariant subspaces are equal.*

Let $\{x_1, x_2\}$ be linearly independent in $\mathcal{X}$; if x_0 is any nonzero vector in $\mathcal{X}$, then (by the Hahn-Banach theorem) there exists a continuous linear functional x^* on

$\mathcal{X}$ such that $x^*(x_0) = 1$. If $\mathcal{M} = \ker(x^*)$, then $\mathcal{X} = \mathcal{M} \oplus \{\lambda x_0 : \lambda \in \mathbb{C}\}$ and the linear map T_0 defined by $T_0|_{\mathcal{M}} = I|_{\mathcal{M}}$, $T_0 x_0 = 0$, belongs to $\mathcal{L}(\mathcal{X})$. Moreover, it is clear that either $T_0 x_1 \neq 0$ or $T_0 x_2 \neq 0$; thus we have the following

Lemma 2. *If $\dim \mathcal{X} > 1$, then $\mathcal{L}(\mathcal{X})$ does not have the property $(\mathcal{P})$.*

Hence, $\mathfrak{A} = \{\lambda I : \lambda \in \mathbb{C}\}$ is a $\mathcal{P}$ -algebra whose commutant $\mathfrak{A}' = \mathcal{L}(\mathcal{X})$ is an s.c.a., but not a $\mathcal{P}$ -algebra ($\dim \mathcal{X} > 1$). Here is an open problem: Assume that $\mathfrak{A}$ is a $\mathcal{P}$ -algebra. Does it imply that $\mathfrak{A}''$ is a $\mathcal{P}$ -algebra too?

The results of [5] can be "specialized" here. In fact, we have

Theorem 2. *Let $\mathfrak{A}$ be an s.c.a. with strictly cyclic vector x_0 . Then every cyclic vector of $\mathfrak{A}$ is actually a strictly cyclic vector. Moreover, the set of all these vectors is open in $\mathcal{X}$ and coincides with $C_{\mathfrak{A}} = \{Ax_0 : A \in \mathfrak{A} \text{ and } A \text{ has a left inverse (mod } \ker S) \text{ in } \mathfrak{A}\}$, where $\ker S$ is the kernel of the map S defined above.*

Proof. If x is a cyclic vector of $\mathfrak{A}$, then $\mathfrak{A}x$ is a dense linear manifold of $\mathcal{X}$ invariant under $\mathfrak{A}$. Since, according to [5, Lemma 2] the only linear manifold of $\mathcal{X}$ satisfying these two conditions is $\mathcal{X}$ itself, we conclude that $\mathfrak{A}x = \mathcal{X}$. In other words, x is a strictly cyclic vector.

Clearly, if $x = Ax_0 \in C_{\mathfrak{A}}$, then there exists $B \in \mathfrak{A}$ such that $BA = I + L$, for some $L \in \ker S$. Therefore, $x_0 = BAx_0 \in \mathfrak{A}x$; hence, $\mathcal{X} = \mathfrak{A}x_0 \subset \mathfrak{A}x$, which implies that x is a strictly cyclic vector of $\mathfrak{A}$.

Conversely, if $x = Ax_0$ is cyclic (and therefore, it is strictly cyclic), then

$$\mathfrak{A}x_0 = x_0 \in \mathcal{X} = \mathfrak{A}x = \{BAx_0 : B \in \mathfrak{A}\}.$$

Therefore, there exists $B \in \mathfrak{A}$ s.t. $(I - BA)x_0 = 0$; i.e., $I - BA = L \in \ker S$. Hence $BA = I - L$, i.e., B is a left inverse of A , modulo $\ker S$. Therefore $x \in C_{\mathfrak{A}}$.

Finally, to see that $C_{\mathfrak{A}}$ is an open subset of $\mathcal{X}$, observe that the mapping S is open and (see, for example, [10, Chapter I]) $S^{-1}(C_{\mathfrak{A}})$ is open in $\mathfrak{A}$.

2. Separated strictly cyclic algebras.

Lemma 3 [10, p. 3]. *Let $\mathfrak{B}$ be a Banach algebra with identity e ; let $\mathfrak{B}_L = \{L_b = \text{left multiplication by } b; b \in \mathfrak{B}\}$ and let $\mathfrak{B}_R$ (right multiplications) be similarly defined (i.e., $b \rightarrow L_b$ and $b \rightarrow R_b$ are the regular left and right, resp., representations of $\mathfrak{B}$). Then:*

(i) $\mathfrak{B}_L$ and $\mathfrak{B}_R$ are separated s.c.a.'s of $\mathcal{L}(\mathfrak{B})$ (i.e., $I \in \mathfrak{B}_L$, $I \in \mathfrak{B}_R$ and $\mathfrak{B}_L, \mathfrak{B}_R$ are strongly closed in $\mathcal{L}(\mathfrak{B})$).

(ii) $\mathfrak{B}'_L = \mathfrak{B}_R$; $\mathfrak{B}'_R = \mathfrak{B}_L$.

Theorem 3. *Let $\mathfrak{A}$ be a separated s.c.a.; then*

(i) $\mathfrak{A}'$ is also a separated s.c.a.;

(ii) $\mathfrak{A}'' = \mathfrak{A}$;

(iii) $\mathfrak{A}$ and $\mathfrak{A}'$ have the property $(\mathcal{P})$.

Proof. Let $\mathfrak{A}$ be a separated s.c.a. with separating strictly cyclic vector x_0 ; then (see [8, Lemma 2.2]) there exists a constant $C \geq 1$ s.t., if $A_x \in \mathfrak{A}$ is the (unique!) operator satisfying $A_x x_0 = x$, then $\|x\| \leq \|A_x\| \leq C\|x\|$.

These two inequalities show that $\mathfrak{A}$ is isomorphic (as a Banach space) with $\mathfrak{X}$. Moreover, if for $x, y \in \mathfrak{X}$ the product $x \cdot y$ is defined by

$$x \cdot y = A_x \cdot A_y x_0,$$

and $\mathfrak{X}$ is re-normed with $\|x\|' = \|A_x\|$, then $\mathfrak{B} = (\mathfrak{X}, +, \cdot, \|\cdot\|')$ is a Banach algebra with identity x_0 , $\|\cdot\|'$ is equivalent to $\|\cdot\|$, $\|x_0\|' = 1$ and $\mathfrak{A}$ can be identified with $\mathfrak{B}_L$. In other words: a separated s.c.a. is essentially the algebra of all "left multiplications" in a Banach algebra with identity. Using this characterization and Lemma 3, we obtain (i) and (ii).

Finally, using (i), (ii) and Theorem 1(i), we obtain (iii).

Thus, if $\mathfrak{A}$ is a separated s.c.a., Theorem 2 applies both to $\mathfrak{A}$ and $\mathfrak{A}'$. Moreover, in this case we also have

Corollary 4. *Let $\mathfrak{A}$ be a separated s.c.a.; then*

(i) *$\mathfrak{A}$ and $\mathfrak{A}'$ have the same separating cyclic vectors (which are actually strictly cyclic vectors for both algebras); this set of vectors is open in $\mathfrak{X}$ and can be identified with*

$$SC = \{Ax_0: A \in \mathfrak{A} \text{ and } A \text{ is invertible}\},$$

where x_0 is an arbitrary separating s.c. vector of $\mathfrak{A}$.

(ii) *Let $A_x \in \mathfrak{A}$, $A'_x \in \mathfrak{A}'$ be the operators defined by*

$$A_x x_0 = A'_x x_0 = x;$$

then $\sigma(A_x) = \sigma(A'_x)$ ($\sigma(T)$ = spectrum of T).

Proof. (i) Identify $\mathfrak{X}$ with a Banach algebra $\mathfrak{B}$ with identity $e = x_0$ as in Theorem 3. Then SC is the set of all invertible elements of $\mathfrak{B}$ and, clearly, every element of SC is a separating strictly cyclic vector of $\mathfrak{B}_L$.

On the other hand, if $f \in \mathfrak{B}$ is a separating s.c. vector of $\mathfrak{B}_L$, then Theorem 2 says that f must have a left inverse in $\mathfrak{B}$. If f is actually invertible, then $f \in SC$, and we are done. Otherwise, there exist two *different* elements, $b, c \in \mathfrak{B}$, s.t. $bf = cf = e$, and therefore

$$0 = (L_b - L_c)f = (L_b - L_c)L_f e = 0 \cdot e,$$

even when $(L_b - L_c) \neq 0$. Hence, f does not separate points of $\mathfrak{B}_L$, contradicting our assumption.

Similarly, we can prove that the set of all separating s.c. vectors of $\mathfrak{A}'$ also coincides with SC .

That SC is open in $\mathfrak{B}$ is immediate (see [10, Chapter I]).

(ii) Using the above identification, consider the following statements:

- (1) $\lambda \in \sigma(L_b)$;
- (2) $L_b - \lambda I$ has no inverse in $\mathfrak{B}_L$;
- (3) $b - \lambda e$ has no inverse in $\mathfrak{B}$;
- (4) $R_b - \lambda I$ has no inverse in $\mathfrak{B}_R$;
- (5) $\lambda \in \sigma(R_b)$.

It is not hard to check that each of these statements is equivalent to the next one. Therefore, $\sigma(L_b) = \sigma(R_b)$.

Remarks. (a) The definition of “algebra with the property ($\mathcal{P}$)” was introduced in [6]. For “strictly cyclic algebras,” see [7], [9], [10].

(b) If $\mathfrak{A}$ is a separated s.c.a., then $\mathfrak{A}'' = \mathfrak{A}$ (by Theorem 3). Therefore, if $A \in \mathfrak{A}$ is invertible in $\mathcal{L}(\mathcal{X})$, then $A^{-1} \in \mathfrak{A}$. However, if A has only a left (or right) inverse $B \in \mathcal{L}(\mathcal{X})$, but A is not invertible, then $B \notin \mathfrak{A}$, in general.

For example (see [8]), if $\mathcal{X}$ is a Hilbert space with orthonormal basis $\{e_n\}$ ($n \geq 0$) and T is the unilateral weighted shift defined by $Te_n = (n+2)/(n+1)e_{n+1}$, then $\mathfrak{A} = \mathfrak{A}_T = \mathfrak{A}'_T$ is a maximal abelian separated s.c.a. and T has a right inverse $L \in \mathcal{L}(\mathcal{X})$; but it is clear (since L does not commute with T !) that $L \notin \mathfrak{A}$.

3. Reflexive bilateral weighted shifts. Examples of separated s.c.a.’s generated by a unilateral weighted shift, or by an invertible bilateral weighted shift and its inverse, can be found in [2], [8], [9]. In particular, the “Donoghue weighted shift” D in the Hilbert space $\mathcal{H}$ (D is defined by $De_n = 2^{-n}e_{n+1}$, $n = 0, 1, 2, \dots$) generates the maximal abelian separated s.c.a. $\mathfrak{A}_D$. This example shows that a $\mathcal{P}$ -algebra need not be a reflexive one. (Furthermore, $\{2^{n(n-1)/2}D^n\}$ $n = 1, 2, \dots$, converges *weakly* to 0; but this sequence does not converge in the strong topology of $\mathcal{L}(\mathcal{X})$. Hence, “strong” cannot be replaced by “weak” in Theorem 1.)

We want to show here that this is no longer the case if D is replaced by a bilateral weighted shift T such that T and T^{-1} generate a separated s.c.a. In fact, our result includes a more general class of shift operators.

In [1], J.A. Deddens remarked that, using the same techniques as in D. Sarason’s paper [11], it is possible to prove that a unilateral weighted shift on a Hilbert space with nontrivial (i.e., other than 0) compression spectrum is reflexive. This result can be extended to a class of bilateral weighted shifts, as follows:

Theorem 4. *Let B be a bilateral weighted shift in a Hilbert space $\mathcal{H}$ with ONB $\{e_n\}$ ($-\infty < n < +\infty$), defined by $Be_n = a_n e_{n+1}$, where $\{a_n\}$ ($-\infty < n < +\infty$) is a bounded sequence of positive reals. Assume that either*

$$I^+ = \liminf_{n \rightarrow +\infty} \left(\prod_{j=0}^{n-1} a_j \right)^{1/n} > 0,$$

or

$$Q^- = \liminf_{n \rightarrow +\infty} \left(\prod_{j=1}^n a_{-j} \right)^{1/n} > 0.$$

Then B is reflexive.

Proof. Let L be a bounded operator in $\mathcal{H}$ and assume that $\mathcal{I}_L \supset \mathcal{I}_B$ (where $\mathcal{I}$ denotes the lattice of invariant subspaces). Assume that $I^+ > 0$ and for each integer m , let $\mathcal{H}_m$ be the closed linear span of $\{e_n : n \geq m\}$. The restriction $B|_{\mathcal{H}_m}$ of B to $\mathcal{H}_m$ is a unilateral weighted shift in $\mathcal{H}_m$ with nontrivial compression spectrum. In fact, the compression spectrum of $B|_{\mathcal{H}_m}$ contains the open disc of radius I^+ (see [2], [4]). It follows from [1] and [12] that

$$(1) \quad L|_{\mathcal{H}_m} = (\text{strong}) \lim_{N \rightarrow +\infty} \sum_{n=0}^N \left(1 - \frac{n}{N+1} \right) c_n (B|_{\mathcal{H}_m})^n$$

where the “Taylor coefficients” c_n ($n \geq 0$) are uniquely determined by L . Moreover, since L maps $\mathcal{H}_m$ into itself for each m , it also follows that L commutes with B on the dense linear manifold $\bigcup \{ \mathcal{H}_m : -\infty < m < +\infty \}$ of $\mathcal{H}$; hence $LB = BL$ and therefore (by a slight modification of the proof given in [3]),

$$(2) \quad L = \lim_{N \rightarrow +\infty} \sum_{n=-N}^{+N} \left(1 - \frac{n}{N+1} \right) c'_n B^n \quad (\text{in the strong topology}),$$

where the “Laurent coefficients” $\{c'_n\}$ ($-\infty < n < +\infty$) are uniquely determined by L . Since $L|_{\mathcal{H}_m} = P_m L P_m = L P_m$, where P_m denotes the orthogonal projection of $\mathcal{H}$ onto $\mathcal{H}_m$, multiplying (2) by P_m on both sides and comparing this expression with (1) it follows that $c'_n = c_n$, for all $n \geq 0$, $c'_n = 0$, for $n < 0$ and $L \in \mathfrak{A}_B$, the strong closure of the polynomials in B . In other words, $\mathfrak{A}_B$ is a reflexive algebra.

If, instead of $I^+ > 0$ we assume that $Q^- > 0$, the result can be proved in the same way, replacing B by B^* and observing (see [1]) that $\mathfrak{A}_B$ is reflexive if and only if $\mathfrak{A}_{B^*}$ is reflexive.

Observe that, in the proof of Theorem 4, we do not use the fact that $\mathcal{I}_L$ contains all the invariant subspaces of B ; in fact, we merely need to assume that $\mathcal{I}_L$ contains every $\mathcal{M} \in \mathcal{I}_B$ such that $\mathcal{M} \subset \mathcal{H}_m$ for some integer m . If B is an invertible bilateral weighted shift, then $I^+(B) > 0$ and $Q^-(B) > 0$ (see [2], [4]) and 0 belongs to the spectrum of $B|_{\mathcal{M}}$ for any invariant subspace $\mathcal{M}$ which is contained in $\mathcal{H}_m$, for some m . Since B is invertible, $\mathcal{M} \notin \mathcal{I}_B^a$, the lattice of analytically invariant subspaces of B (see [6, §2]). Thus, we obtain the following

Corollary 5. *An invertible bilateral weighted shift B in a Hilbert space $\mathcal{H}$ is always reflexive. Moreover, if L is any bounded linear operator in $\mathcal{H}$ such that $\mathcal{I}_L \supset \mathcal{I}_B \setminus \mathcal{I}_B^a$, then $L \in \mathfrak{A}_B$.*

Finally, we shall provide a counterexample for the following question:

Let $T \in \mathcal{L}(\mathcal{X})$ be a reflexive operator, $\mathcal{M} \in \mathcal{J}'_T$, and $R = T|_{\mathcal{M}}$; is R a reflexive operator on $\mathcal{M}$? ($\mathcal{J}'_T$ = lattice of $\mathfrak{A}'_T$.)

In the case when $\mathcal{X}$ is finite dimensional, the answer is yes, as it follows from [1, Theorem 2]. This is no longer true if $\mathcal{X}$ is infinite dimensional; in fact, if B is defined as in Theorem 4, with $a_n = 1$, for $n \leq 0$, $a_n = 2^{-n}$, for $n > 0$, then $I^+ = 0$, $Q^- = 1$ and B is reflexive. Since B is not invertible, $\mathfrak{A}'_B = \mathfrak{A}_B$ (see [3]) and therefore $\mathcal{J}'_B = \mathcal{J}_B$. However $B|_{\mathcal{M}_0}$ is a "Donoghue unilateral weighted shift" [2] and therefore, it is not reflexive (see [1]).

Similarly, it is possible to show that if T is reflexive, $\mathcal{M} \in \mathcal{J}'_T$ and $\bar{T}$ is the operator induced by T on $\mathcal{X}/\mathcal{M}$, then $\bar{T}$ is not necessarily reflexive.

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